QUALITATIVE INVESTIGATION OF AN EQUATION OF THE THEORY OF PHASE AUTOMATIC FREQUENCY CONTROL

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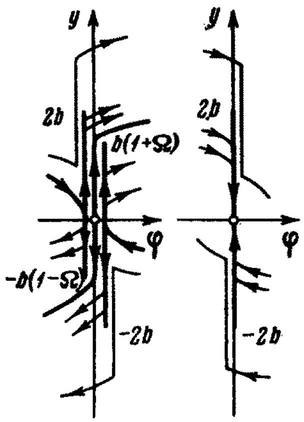
We use the point transformation method to investigate on a cylindrical phase space a piecewise-linear second-order dynamic system whose representing point experiences jumps at the straight-line "seams" of the cylindrical space.

Equation
$$\varphi^{\cdot\cdot} + h\,[1 + bF'(\varphi)]\,\varphi^{\cdot} + F(\varphi) = \Omega \quad (b > 0,\ h > 0,\ 0 \leqslant \Omega < 1) \tag{1}$$
where $F(\varphi)$ is a 2π-periodic function such that
$$F(\varphi) = \begin{cases} -1 & \text{when } -\pi < \varphi < 0 \\ 1 & \text{when } \quad 0 < \varphi < \pi \end{cases} \tag{2}$$

describes the dynamics of a phase automatic frequency control (afc) system with a proportional integrating filter [1 and 2] and a rectangular characteristic of the phase detector [3]. It is necessary to take into account that this equation becomes meaningless at the points of discontinuity of the characteristic $F(\varphi)$.



a) b)

Fig. 1

The phase space of the system
$$\varphi^{\cdot} = y, \qquad y^{\cdot} = \Omega - h\,[1 + bF'(\varphi)]\,y - F(\varphi) \tag{3}$$
corresponding to (1) is a cylinder. Introducing the new variables t° and y° which from now on we denote by t and y
$$t^\circ = ht, \quad y^\circ = y \,/\, h \tag{4}$$
we can reduce the system to the form
$$\varphi^{\cdot} = y, \qquad y^{\cdot} = [\Omega - F(\varphi)] \,/\, h^2 - [1 + bF'(\varphi)]\,y \tag{5}$$
Introducing the notation
$$a = (1 + \Omega) \,/\, h^2\pi, \qquad c = (1 - \Omega) \,/\, h^2\pi \leqslant a \tag{6}$$
we can write system (5) as
$$\varphi^{\cdot} = y,\ y^{\cdot} = a\pi - y \quad (-\pi < \varphi < 0) \tag{7}$$
$$\varphi^{\cdot} = y,\ y^{\cdot} = -c\pi - y \quad (0 < \varphi < \pi) \tag{8}$$

Systems (7) and (8) enable us to follow the motion of the representing point up to the instant when it reaches one of the straight lines $\varphi = 0$ or $\varphi = \pi$ on which system (5) is not defined.

The subsequent motion of the representing point requires further definition. We must specify how long it remains on the straight line, the manner of its motion along this line, the point at which it leaves the line and which of systems (7) and (8) describes its further motion. We shall utilize the additional definition constructed in [4]. (When using the formulas (*) of [4], we must take into account the change of scale of t and y defined by (4)).

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*) The expression $r = 2b,\ h > 0$ appearing in the fourth line from top on p. 756 of [4] should read $= 2bh > 0$.

Figure 1a shows the pattern of the additionally defined motions along the straight line $\varphi = \pi$. The overlapping trajectories have been resolved on the φ-axis for clarity. The representing point of system (5) arriving at the point (π, y) where $y > 0$ experiences an instantaneous jump of magnitude 2b upwards, along the line $\varphi = \pi$ and continues to move in $\varphi > \pi$ according to (7). If the representing point arrives at the point $(\pi, 0)$ from the region $y > 0$, then its further behavior is not uniquely defined. It remains at the point $(\pi, 0)$ for some time and then either jumps to one of the points on the segment $\varphi = \pi$, $b(1 + \Omega) < y \leqslant 2b$ and continues to move in $\varphi > \pi$ according to the system (7), or jumps to a point on the segment $\varphi = \pi$, $-b(1 - \Omega) < y \leqslant 0$ and continues to move in $\varphi < \pi$ according to (8).

The representing point moving in the lower half-cylinder and arriving at the line $\varphi = \pi$ behaves similarly (Fig. 1a). The point $(\pi, 0)$ represents an equilibrium state of the saddle type and the trajectories passing through the points $[\pi, b(1 + \Omega)]$ and $[\pi, -b(1 - \Omega)]$ are its separatrices. The trajectories entering the point $(\pi, 0)$ play the role of the other two separatrices.

Figure 1b shows the pattern of additionally defined motions on the straight line $\varphi = 0$. The representing point, having arrived at the point $(0, y)$ where $y > 2b$ experiences an instantaneous jump of length $2b$ down the y-axis and then continues to move in $\varphi > 0$ according to system (8). If the representative point arrives at the point $(0, y)$ where $0 < y \leqslant 2b$, then it jumps into the point $(0, 0)$ and remains there idefinitely.

The behavior of the representing point moving in the lower half-cylinder and arriving at the line $\varphi = 0$ is analogous to the previous case (Fig. 1b). Point $(0, 0)$ represents an equilibrium state analogous of the stable-node type.

From now on we shall assume that (5) is the additionally defined system.

Since systems (7) and (8) have no equilibrium states, additionally defined system (5) also has no equilibrium states apart from the points $(0, 0)$ and $(\pi, 0)$.

It can be shown that system (5) has neither limit cycles encompassing the lower half-cylinder $(y < 0)$, nor limit cycles not encompassing the cylinder.

We shall now establish the existence and find the number and stability of the limit cycles encompassing the upper half-cylinder. Let us take on the straight line $\varphi = -\pi$ the half-line Z: $\varphi = -\pi$, $y = z \geqslant 0$ and consider a trajectory originating at some point z of this half-line. If z is sufficiently large, then the half-trajectory in question will go around the cylinder and re-enter Z at a different point z_1. This means that the set of similar trajectories effects a point transformation of the half-line Z into itself (we denote this transformation by S), and that its points z and z_1 are in one-to-one correspondence. The transformation S results from the two transformations T and T_1 applied consecutively. Here by T we mean the travel of the representing point from the half-line Z to the half-line V: $\varphi = 0$, $y = v > 0$, followed by a jump along the latter (the jumps referred to here do not terminate in the equilibrium state). T_1 denotes the passage of the representing point from V to Z, followed by an additionally defined motion along the latter. Integration of (7) and (8) yields parameteric equations of the mapping function relating the transformations T and T_1. For T these equations are

$$\frac{z}{\pi} = a + \frac{1 - a\tau}{1 - e^{-\tau}}, \quad \frac{v}{\pi} = a + \frac{1 - a\tau}{e^{\tau} - 1} - \frac{2b}{\pi} \tag{9}$$

and the parameter τ is the time of travel of the representing point from Z to V. This time must be varied from zero to a value at which z or v becomes zero. The derivatives of (9) are of the form

$$\frac{dz}{dv} = \frac{e^{\tau}(v+2b)}{z} > 0, \quad \frac{d^2z}{dv^2} = -a\pi e^{\tau}(e^{\tau}-1)\frac{z+v+2b}{z^3} < 0 \qquad (10)$$

and the asymptotic representation of (9) is

$$z = v + 2b + \pi \qquad (11)$$

Figure 2 depicts the trajectories which effect the transformation T_1. If the points v lie on V above the trajectory of (8) which passes through $(\pi, 0)$, then the mapping function for T_1 is

$$\frac{z_1}{\pi} = -c + \frac{1+c\theta}{e^{\theta}-1} + \frac{2b}{\pi}, \quad \frac{v}{\pi} = -c + \frac{1+c\theta}{1-e^{-\theta}} \qquad (12)$$

The parameter θ appearing here denotes the time of travel of the representing point from V to Z. This time must be varied from zero to a value θ_0 such that $z_1 = 2b$. The derivatives of (12) are of the form

$$\frac{dz_1}{dv} = \frac{ve^{-\theta}}{z_1-2b} > 0 \qquad \frac{d^2z_1}{dv^2} = -c\pi e^{-\theta}(1-e^{-\theta})\frac{(z_1+v-2b)}{(z_1-2b)^3} < 0 \qquad (13)$$

and the asymptotic representation of (12) is

$$z_1 = v + 2b - \pi \qquad (14)$$

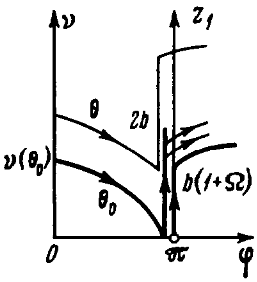


Fig. 2

The transformation T_1 puts the point $v(\theta_0)$ and the segment $b(1+\Omega) < z_1 \leqslant 2b$ on one-to-one correspondence, so that the mapping function for these v and z_1 is represented on the vz-plane by the segment

$$v = v(\theta_0), \quad b(1+\Omega) < z_1 \leqslant 2b \qquad (15)$$

To find the stationary points of the transformation S we must find the common points of the curves $z = z(v)$ (9) and $z_1 = z_1(v)$ (12), (15). If b is sufficiently large, then the point of intersection of the curve $z = z(v)$ with the z-axis lies above the point of intersection of the asymptotic curve $z_1 = z_1(v)$ with the z-axis, and the curves have no common points (see Fig. 3 where the curves $z = z(v)$ and $z_1 = z_1(v)$ are denoted by z and z_1, respectively). Let us decrease b, keeping the remaining parameters constant. For $v = \text{const}$ we have

$$d(z-z_1)/db = 2(dz/dv - 1) > 0 \qquad (16)$$

since, by (10) and (11), $dz/dv > 1$). This implies that the curves approach each other, and either that common points appear for sufficiently small b, or that b vanishes without any common points appearing. We can see which values of a and c correspond to one or the other situation by considering the curves z and z_1 at $b = 0$ (Fig. 4). When the values of a and c correspond to Fig. 4a we have the first case; Fig. 4b corresponds to the second case. The curve separating these two cases on the ac-plane is defined by the following conditions:

$$z(\tau) = 0, \quad z_1(\theta) = 0, \quad v(\tau) = v(\theta) \qquad (17)$$

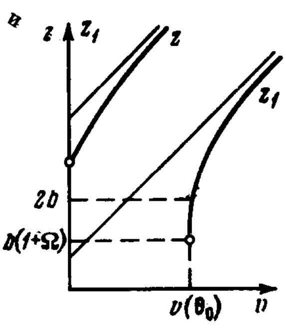
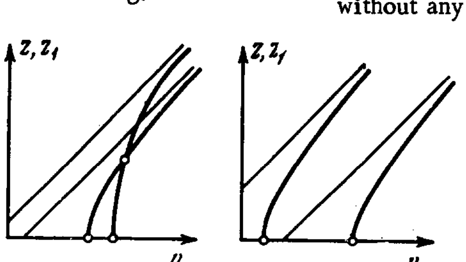


Fig. 3



a) b)

Fig. 4

or, in expanded form,

$$c^{-1} = \exp\theta - \theta - 1 \equiv \varphi(\theta), \quad a^{-1} = \varphi(-\tau) \quad \tau / \varphi(-\tau) - \theta / \varphi(\theta) = 2 \qquad (18)$$

The function represented by the curve is an increasing one; it lies in the region $c < a$ and passes through the point $c = 0$, $a = a_*$, where $\exp(-2/a_*) + 1/a_* - 1 = 0$. The region corresponding to Fig. 4a lies between the curve (18) and the a-axis. If the point (a, c) lies outside this region, then the curves z and z_1 have no common points and system (5) has no limit cycles for any b.

Figure 5a shows the corresponding phase trajectories of the system (the scale of φ has been made nonuniform to bring both equilibrium states to the front half of the cylinder; had it been kept uniform, one of the states would be hidden behind the cylinder). Let us now suppose that the point (a, c) lies within the region bounded by the curve (18) and the a-axis. With sufficiently large b, we have the pattern of phase trajectories shown in Fig. 5a.

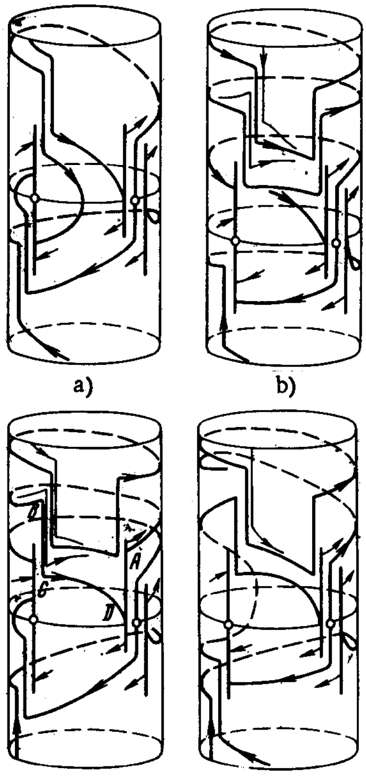


Fig. 5

With decreasing b there is an instant when the curves make a single point contact and a conditionally stable limit cycle appears on the phase cylinder. As the value of b continues to decrease, the point of contact becomes two points of intersection and the conditionally stable limit cycle becomes two coarse limit cycles, the upper one stable and the lower one – unstable. Figure 5b shows the corresponding pattern of the phase trajectories. After this, the point of intersection corresponding to the stable limit cycle passes to the vertical part of the curve z_1, and the stable limit cycle becomes the closed contour $ABCDA$ (Fig. 5c). This cycle separates the regions of attraction of the stable limit cycle and the state of stable equilibrium on the phase cylinder in the same way as the unstable limit cycle.

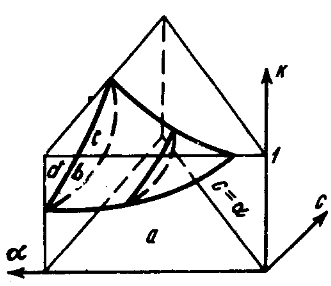


Fig. 6

With further decreases in b, the distance between the point in question and the end point of z_1 diminishes, and the point finally vanishes. The contour $ABCDA$ merges with the loop-shaped separatrix of the state of equilibrium $(\pi, 0)$. Continued decreases in the value of b to zero does not result in formation of new bifurcations. The system has a single stable limit cycle and the corresponding pattern of phase trajectories is shown in Fig. 5d.

The above description is based on the fact that the curves (9) and (12) can intersect at not more than two points (see Addendum).

The surface in the parametetric space on which the conditionally stable limit cycle is formed is defined by the conditions

$$v(\tau) = v(\theta), \qquad z(\tau) = z_1(\theta), \qquad dz / dv = dz_1 / dv \qquad (19)$$

Let us rewrite these equations in expanded form, replacing the first equation by the difference between the first and the second one. Then setting

$$k = 1 - 2b / \pi (a + c) = 1 - bh^2 \qquad (20)$$

we obtain

$$a\tau - c\theta = 2, \quad k(a + c) = \frac{c\theta + 1}{e^{\theta} - 1} + \frac{a\tau - 1}{1 - e^{-\tau}}, \quad \frac{1 + a\varphi(\tau)}{1 - a\varphi(-\tau)} = \frac{1 + c\varphi(-\theta)}{1 - c\varphi(\theta)} > 0 \qquad (21)$$

Surface (21) (the lower surface on Fig. 6) intersects the plane $k = 1$ $(b = 0)$ along the curve (18), and the plane $c = 0$ along the curve

$$ak = (1 - e^{-2/a})^{-1} \qquad (22)$$

and has the asymptotic plane $k = 1 / 2$. Making use of the rotation of the vector field of system (8) with varying c, we can show that the intersection of this surface with the plane $a = \text{const}$ represents a graph of an increasing function (Eqs. (21) can be used to construct the numerical dependence of the pull-in range [2] on the parameters of the system).

The conditions

$$z(\tau) = b(1 + \Omega), \qquad v(\tau) = v(\theta), \qquad z_1(\theta) = 2b \qquad (23)$$

define the surface in the parameter space corresponding to the formation of the separatrix loop.

Let us substract the first equation of (23) from the sum of the remaining two and substitute the resulting equation in place of the second equation of (23). This gives us

$$a^{-1} = \tau - k(1 - e^{-\tau}), a\tau - c\theta = 2 + c(1 - k), \quad c^{-1} = \varphi(\theta) \qquad (24)$$

We can construct the surface (24) easily, using the intersections with $c = \text{const}$. Its lines of intersection with the planes $k = 1$ and $c = 0$ are identical to those of surface (21); its asymptotic cylinder is

$$2k = 2e^{\theta} - \theta - \sqrt{(2e^{\theta} - \theta)^2 - 4e^{\theta}}, \quad c^{-1} = \varphi(\theta) \qquad (25)$$

The intersections of the surface with the planes $a = \text{const}$ and $c = \text{const}$ are graphs of monotonic functions.

Figure 6 shows a schematic breakdown of the parametric space of the system under investigation into regions in which the phase trajectories have distinct qualitative structures. The letters a, b, c, d indicate the regions corresponding to the respective patterns of Fig. 5.

Note. When the dynamics of a device is described by system (3), then the stabilizing processes begin on a certain circle surrounding the phase cylinder ([5], p. 183). It follows that the device will operate not only when its parameters are such that the system has no limit cycles, but also when the system has two limit cycles, provided the circle of initial values lies below the unstable limit cycle.

Addendum. We shall show that the curves (9) and (12) can have at most two points of intersection. Let us find the difference of the derivatives at these points

$$\frac{dz}{dv} - \frac{dz_1}{dv} = \frac{v + 2b}{z} \frac{z - a\pi}{v + 2b - a\pi} - \frac{v}{z - 2b} \frac{z - 2b + c\pi}{v + c\pi} \equiv f(v, z)$$

where the quantities τ and θ have been eliminated by means of (9) and (12). Function $f(v, z)$ changes sign at the straight lines $z = 2b$, $v = a\pi - 2b$, $z = v + 2b$ and at the hyperbola

$$(a + c) zv + 2bcz - 2bav - 2\pi bac = 0$$

Inspecting the phase pattern, we find that for the limit cycles we have $v < a\pi - 2b$ and $z > 2b$. Within the region defined by these inequalities the function $f(v, z)$ moving along the curve (9) changes its sign once only, namely at the point of intersection of (9) with the hyperbola (the line $z = v + 2b$ does not intersect (9) within this region). If the curves (9) and (12) intersected each other at more than two points, then the function $f(v, z)$ would change sign more than once on (9).

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AN OPTIMAL TERMINAL CONTROL PROBLEM

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The problem of choosing a law of time variation of controlling forces of bounded absolute value which ensure a minimal deviation measure at the end of the trajectory and a minimal control measure is investigated for linear systems with a fixed time of motion. It shows that a unique optimal trajectory and a unique control exist for this optimal terminal control problem. The possibility of using the Pontriagin maximum principle to solve this problem is demonstrated and the practical difficiulties of such an approach are pointed out. These difficulties can be overcome by means of the proposed approximate method for solving the two-point boundary value problem arising from the application of the maximum principle. A procedure for the practical realization of the above method on a computer is described.

1. Formulation of the problem. Let the motion of some system be described by the following differential equations with variable coefficients:

$$\frac{dx_\nu}{dt} = \sum_{\nu=1}^{n} a_{\nu\mu}(t)\, x_\mu + \sum_{\rho=1}^{m} b_{\nu\rho}(t)\, u_\rho(t) + f_\nu(t), \qquad x_\nu(t_0) = z_\nu^\circ \quad (\nu = 1, \ldots, n) \qquad (1.1)$$

Here x_ν are the phase coordinates of the system in question; $a_{\nu\mu}(t)$ and $b_{\nu\rho}(t)$ are the system parameters varying continuously with time; $f_\nu(t)$ are the prescribed external forces; $u_\rho(t)$ are the controlling forces of bounded absolute value whose law of variation